Intermittency in Drift-Wave Turbulence: Structure of the Momentum Flux Probability Distribution Function

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We analytically compute the probability distribution function (PDF) of the local Reynolds stress (R) for forced Hasegawa-Mima turbulence. With the assumption that the PDF tail is due to an instanton with the spatial form given by the modon solution, the tail of the PDF of R is found to be a stretched, non-Gaussian exponential, with the specific form $\exp[-cR^{3/2}]$ (c is a constant). We relate the temporal localization of the instanton to the degree of "burstiness" of the momentum transport event.

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Prediction of particle and heat transport is essential to obtaining controlled magnetic fusion. In particular, large transport events of substantial amplitude, even if rare, can be dangerous for confinement, on account of their strength. For instance, the breaking of gyro-Bohm scaling observed in large tokamaks as well as in recent numerical simulations [1] indicate that avalanche-like events of a large amplitude can be crucial in the transport (e.g., in the L mode). These events are associated with coherent structures such as streamers or blobs and are often bursty in time, thus leading to non-Gaussian statistics in probability distribution function (PDF) tails. Therefore, transport models must confront the problem of prediction of rare, large transport events. The focus of this Letter is thus on the prediction of PDF tails arising from these bursty events associated with coherent structures.

Theoretically, one can obtain PDF tails in the long time limit by taking an ensemble average. For instance, let us imagine constructing tails of flux PDF owing to fluid velocity at $t = 0$, by turning on an external forcing at $t = -\infty$ when there is no fluid motion. We further assume that at $t = 0$, there are coherent structures that are responsible for PDF tails. Between the initial ($t = -\infty$) and final times ($t = 0$), coherent structures would appear at arbitrary times, followed by their destruction. As PDF tails are determined by the amplitude of these coherent structures at $t = 0$, we can relate PDF tails to a joint probability of creation and decay of various coherent structures (or, the transition amplitude between the state with no fluid motion and that with coherent structures). Here, the coherent structure can be viewed more generally as the spatial pattern associated with an "empirical eigenfunction" of the turbulence. While a weighted sum over these different coherent structures would eventually be necessary to obtain the true PDF tail, for simplicity, we shall assume the presence of only one single coherent structure [2] in the following discussion and call the creation and destruction processes of this coherent structure an instanton and an anti-instanton, respectively. Note that the physical meaning of instanton/anti-instanton employed here is slightly different from that used in quantum mechanics [3].

An instanton solution for a dynamical variable u in classical fluid problems [2] is a nonperturbative solution in the form of $u(\mathbf{x}, t) = F(t)u_0(\mathbf{x})$ with $F(t \to -\infty) = 0$. Here, $u_0(\mathbf{x})$ denotes the spatial form of a coherent structure (i.e., more specifically, an exact solution of the time-independent dynamical equation in a certain frame) and $F(t)$ is a temporally localized amplitude, capturing its creation process. For instance, u_0 can be an exact solution to a homogeneous nonlinear equation, in which case $F(t)$ represents the excitation of u_0 from the initial state with $u = 0$ by an external forcing. As we shall see shortly, $F(t)$ is to be determined by a saddle-point method applied to an effective action expression for the PDF, which can, in general, be given by a path integral [2,4,5]. Thus, the PDF estimate is intrinsically nonperturbative. The opposite can apply to an anti-instanton, which starts with a finite (initial) value of $F(t)$ and ends with a vanishing (final) value; $F(t)$ in this case represents the destruction of a coherent structure. In principle, one should incorporate the contribution to PDF tails from multi-instantons, a time sequence like instanton, anti-instanton, instanton, for example.

In this Letter, we compute the PDF tail of local Reynolds stress R (momentum flux) for forced Hasegawa-Mima turbulence [6] in order to study momentum transport. Note that Hasegawa-Mima turbulence is the simplest paradigm of drift-wave turbulence. While heat and/or particle transport events are ultimately of greater interest, the simplicity of the Hasegawa-Mima model forces us to consider only momentum transport events here. A "momentum transport event" may be thought of as a localized burst of shear generation such as that which occurs when the $L \to H$ transition is triggered. Note that the generation of shear flow has been extensively studied in many astrophysics, geophysical, and laboratory settings, with examples ranging from the Jovian belt-type flow, the rapid rotation of the Venusian atmosphere, to the zonal winds on major planets Jupiter and Saturn. In the absence of dissipation and external forcing, the Hasegawa-Mima equation allows an exact solution, known as a modon [7], which is a bipolar vortex soliton. As this is the only exact solution that is available to us, we consider coherent structures (which contribute to

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PDF tails) to be modons, for simplicity. Furthermore, to simplify the analysis, we take into account only a single instanton [2]. The key idea is then to view the PDF tails as a transition amplitude from an initial state, with no fluid motion, to final states with different values of R due to a modon in the long time limit.

A forced Hasegawa-Mima equation [6] takes the following form:

$$(1 - \nabla^2)\partial_t \phi + v_* \partial_y \phi - \mathbf{v} \cdot \nabla\nabla^2 \phi = f\,. \qquad (1)$$

Here, the notation is standard; x and y denote local radial and poloidal directions, respectively; $v_* = \rho_s^2 \Omega_i / L_n$ is the drift velocity due to radial density gradient; $L_n = -(\partial_x n_0/n_0)^{-1}$ is the (background) density length scale; ϕ, $\mathbf{v} = -\nabla \times \phi \hat{z}$, and f are, respectively, electric potential, $\mathbf{E} \times \mathbf{B}$ advection velocity, and external forcing. Note that Eq. (1) is nondimensionalized by measuring the length, velocity, and ϕ in units of ρ_s, c_s, and T_e/e.

To keep the analysis tractable, we shall take the statistics of the forcing to be Gaussian with white noise in time as follows:

$$\langle f(\mathbf{x}, t) f(\mathbf{x}', t') \rangle = \delta(t - t') \kappa(\mathbf{x} - \mathbf{x}')\,, \qquad (2)$$

and $\langle f \rangle = 0$. By exploiting the Gaussian statistics of the forcing [5], the PDF for the local Reynolds stress $v_x v_y(\mathbf{x}_0) = -\partial_x \phi \partial_y \phi(\mathbf{x}_0)$ can be expressed in terms of a path integral [2,4]:

$$P(R; \mathbf{x}_0) = \langle \delta(v_x v_y|_{\mathbf{x}_0} - R) \rangle = \int d\lambda e^{i\lambda R} I_\lambda\,, \qquad (3)$$

where

$$I_\lambda = \int \mathcal{D}\phi \mathcal{D}\overline{\phi}\, e^{-S_\lambda}.$$

In Eq. (3), the angular brackets denote the average over the random forcing f, and S_λ is the effective action given by

$$\begin{aligned} S_\lambda = -i \int d^2x\, dt \overline{\phi}[(1 - \nabla^2)\partial_t \phi + v_* \partial_y \phi - \mathbf{v} \cdot \nabla\nabla^2 \phi] \\ + \frac{1}{2} \int d^2x d^2x' dt\, \overline{\phi}(\mathbf{x}) \kappa(\mathbf{x} - \mathbf{x}') \overline{\phi}(\mathbf{x}') \\ + i\lambda \int d^2x dt\, (-\partial_x \phi \partial_y \phi) \delta(t) \delta(\mathbf{x} - \mathbf{x}_0)\,. \qquad (4) \end{aligned}$$

As we are interested in PDF tails for a large R, we can evaluate the λ integral in Eq. (3) by a saddle-point method, once I_λ is computed. Furthermore, the leading order term in I_λ can be evaluated by a saddle-point method provided that λ is a large parameter. In fact, λ can be taken as a large parameter for $n > 1$, when $I_\lambda \sim \exp[-S_\lambda(0)]$ with $S_\lambda(0) = iC\lambda^n$, since the saddle-point solution for the λ integral satisfies $\lambda \propto R^{1/(n-1)}$. Here, $S_\lambda(0)$ is the saddle-point action, and C and n are constants. A saddle-point solution ϕ for large R and λ with the initial condition $\phi(t \to -\infty) = 0$ constitutes an instanton, as mentioned previously.

Expecting a coherent structure, contributing to PDF tails, to be a modon created by the external forcing, we make the ansatz that the saddle-point solution (instanton) is a temporally localized modon, i.e.,

$$\phi(\mathbf{x}, t) = \psi(\mathbf{x}, t) F(t)\,, \qquad (5)$$

where $\psi(\mathbf{x}, t) = \psi(x, y - Ut)$ is a modon solution given by [7]

$$\begin{aligned} \psi_< &= [c_1 J_1(kr) + (\beta - k^2 U) r/k^2] \cos\theta\,, \\ \psi_> &= c_2 K_1(pr) \cos\theta\,. \end{aligned} \qquad (6)$$

Here $\psi_< = \psi(r < a)$ and $\psi_> = \psi(r > a)$; $r = \sqrt{x^2 + y'^2}$, $\tan\theta = y'/x$, $y' = y - Ut$, $\beta = v_* - U$, $p^2 = -\beta/U$, $c_1 = -\beta a/k^2 J_1(ka)$, $c_2 = -Ua/K_1(pa)$, and $J_1'(ka)/J_1(ka) = (1 + k^2/p^2)/ka - kK_1'(pa)/pK_1(pa)$; U is the velocity of a modon; a is the size of the core region; J_1 and K_1 are the first Bessel and the second modified Bessel functions. In the following analysis, the size of a modon a shall be taken to be a fixed parameter. As noted previously, PDF tails are then interpreted as the transition amplitude going from the state with no fluid motion to final states with different values of local Reynolds stresses due to different amplitude of a modon of a given size a. The time variation of ϕ, i.e., $F(t)$ in Eq. (5), representing the excitation of a modon by an external forcing, can be associated with the degree of "burstiness" of an event.

By using Eq. (5), we can perform the spatial integral in S_λ, in Eq. (4), as follows. First, we assume that κ in Eq. (2) is approximately parabolic for $|\mathbf{x} - \mathbf{x}'| < L$ with the following form:

$$\kappa(\mathbf{x} - \mathbf{x}') = \kappa_0 J_0(k_f |\mathbf{x} - \mathbf{x}'|)\,, \qquad (7)$$

and vanish for $|\mathbf{x} - \mathbf{x}'| > L$. Here $L \lesssim \alpha_{01}/k_f$ with α_{01} being the first zero of J_0. Note that the choice of J_0 is just for computational convenience. Second, we expand the conjugate variable in terms of Bessel and Fourier series:

$$\begin{aligned} \overline{\phi}_< &= \sum_{m,n} J_m\left(\frac{\alpha_{mn}}{a} r\right)[a_{mn}(t) \sin m\theta + b_{mn}(t) \cos m\theta]\,, \\ \overline{\phi}_> &= \sum_{m,n} K_m(q_{mn} r)[\overline{a}_{mn}(t) \sin m\theta + \overline{b}_{mn}(t) \cos m\theta]\,, \end{aligned} \qquad (8)$$

where $\overline{\phi}_< = \overline{\phi}(r < a, \theta, t)$ and $\overline{\phi}_> = \overline{\phi}(r > a, \theta, t)$; $a_{mn}(t)$, $b_{mn}(t)$, $\overline{a}_{mn}(t)$, and $\overline{b}_{mn}(t)$ are unknown functions of time, which are to be determined by solving saddle-point equations.

By using Eqs. (5)–(8) in (4), S_λ reduces to an integral with respect to time only:

$$S_\lambda = -i \int dt \sum_n [\dot{F}(\overline{A}_n \overline{b}_{1n} + A_n b_{1n}) + F(F-1) B_n a_{2n}] + \kappa_0 \int dt \sum_{m,n} [(D_n b_{1n} + \overline{D}_n \overline{b}_{1n})(D_m b_{1m} + \overline{D}_m \overline{b}_{1m}) + E_m E_n a_{2m} a_{2n}] + i\lambda \int dt\, F^2 \xi_0 \delta(t). \quad (9)$$

Here, $\xi_0 = -\partial_x \psi \partial_y \psi(\mathbf{x}_0)$, $A_n = c_1(1+k^2) \int_0^a dr \times rJ_1(kr)J_1(z_{1n})/2 + \alpha \int_0^a dr\, r^2 J_1(z_{1n})/2k^2$, $B_n = -k\alpha \times \int_0^a dr\, rJ_2(kr)J_2(z_{2n})/4$, $D_n = \int_0^a dr\, rJ_1(k_f r)J_1(z_{1n})/2$, $E_n = \int_0^a dr\, rJ_2(k_f r)J_2(z_{2n})/2$, $\overline{A}_n = c_2(1-p^2) \times \int_a^L dr\, rK_1(q_{1n}r)K_1(pr)/2$, and $\overline{D}_n = \int_a^L dr\, rK_1(q_{1n}r) \times J_1(k_f r)/2$; $z_{1n} = \alpha_{1n} r/a$ and $z_{2n} = \alpha_{2n} r/a$; α_{in} is the nth zero of J_i [i.e., $J_i(\alpha_{in}) = 0$]; q_{1n} is a constant; $\alpha = \beta - k^2 U$. Note that A_n, D_n, $\overline{A}_n$, and $\overline{D}_n$ here originate from the terms involving $\cos\theta$, while B_n and E_n arise from those with $\sin 2\theta$. Note also that the coefficients A_n, $\overline{A}_n$, and B_n involve the projection of the conjugate variable onto the modon, while D_n, $\overline{D}_n$, and E_n contain the projection of the forcing onto the conjugate variable.

By minimizing S_λ (4) with respect to independent variables $F(t)$, $a_{2n}(t)$, $b_{1n}(t)$, $\overline{a}_{2n}(t)$, and $\overline{b}_{1n}(t)$, we obtain the following saddle-point equations:

$$-iA_n \partial_t F + 2\kappa_0 \sum_m (D_m b_{1m} + \overline{D}_m \overline{b}_{1m}) D_n = 0, \quad (10)$$

$$-iB_n F(F-1) + 2\kappa_0 \sum_m E_n E_m a_{2m} = 0, \quad (11)$$

$$-i\overline{A}_n \partial_t F + 2\kappa_0 \sum_m (D_m b_{1m} + \overline{D}_m \overline{b}_{1m}) \overline{D}_n = 0, \quad (12)$$

$$\sum_n [A_n \partial_t b_{1n} + \overline{A}_n \partial_t \overline{b}_{1n} - B_n(2F-1) a_{2n}] = -2\lambda F(t) \delta(t) \xi_0. \quad (13)$$

Equation (13) implies that b_{1n} and $\overline{b}_{1n}$ have a discontinuity at $t = 0$, since the physical quantity $F(t)$ is a smoothly varying function of time. Furthermore, as conjugate variables propagate backwards in time in the presence of dissipation [2], $b_{1n} = 0$, $\overline{b}_{1n} = 0$, and $a_{2n} = 0$ for $t \geq 0$. We thus integrate Eq. (13) for a small time interval $t \in [-\epsilon, 0]$ ($\epsilon \ll 1$) to obtain the relation at $t = -\epsilon$,

$$\sum_n [A_n b_{1n} + \overline{A}_n \overline{b}_{1n}] = 2\lambda F_0 \xi_0, \quad (14)$$

where $F_0 = F(t = 0)$. Note that the discontinuities in b_{1n} and $\overline{b}_{1n}$ at $t = 0$ are directly related to the nonvanishing value of F_0. For $t < 0$, the coupled equations (10)–(13) yield an equation for F as

$$\partial_{tt} F - \gamma(F^2 - F)(2F - 1) = 0, \quad (15)$$

where $\gamma = \sum_m B_m B_m / (Q \sum_n E_n E_n)$ and $Q = \sum_m A_m A_m / \sum_n D_n D_n = \sum_m \overline{A}_m \overline{A}_m / \sum_n \overline{D}_n \overline{D}_n$. The solution to Eq. (15), with the boundary conditions $F(t = 0) = F_0$ and $F(t \to -\infty) = 0$, is easily found to be

$$F(t) = \frac{1}{1 - \frac{F_0 - 1}{F_0} \exp\{-\sqrt{\gamma}\, t\}}, \quad (16)$$

while the value of F_0 is determined by Eqs. (10), (12), and (14) as

$$F_0 = 1 + \frac{i4\kappa_0 \lambda}{\sqrt{\gamma}\, Q} \xi_0 .$$

As can be seen from Eq. (16), the instanton is localized within a time interval proportional to $1/\sqrt{\gamma}$. By adjusting the forcing, the localization time of the instanton can be chosen to be shorter than the viscous time scale, thereby justifying the neglect of viscosity in the present analysis. Note that Eqs. (10)–(13) indicate a nonvanishing projection of the forcing onto the modon is necessary for the existence of a nontrivial solution for F. Thus, the (spatial) "overlap" between the forcing and modon is critical for the generation of the modon. This projection is likely to be maximized by choosing the characteristic scale of the forcing to be comparable to that of a modon, i.e., when $k_f \sim k$. In more general terms, which coherent structure is likely to be generated is determined by the nature of the forcing, with different forcings giving rise to different manifestations of intermittency.

The instanton solution (16), with the help of Eqs. (10)–(13), then gives us the saddle-point action to leading order in λ, as

$$S_\lambda(0) \simeq -\frac{i}{3} h \lambda^3,$$

where $h = \xi_0^3 q^2$ and $q = |4\kappa_0/(\sqrt{\gamma}\, Q)|$. The previous equation justifies the assumption of large λ since $S_\lambda(0) \propto \lambda^3$ with $n = 3$. Finally, the PDF tails for local Reynolds stress R can easily be computed by applying the saddle-point method for the remaining λ integral in Eq. (3). It turns out that the PDF tail is physically meaningful only for $R/\xi_0 > 0$, because an instanton with the opposite sign of R to the modon is unlikely to be excited. For $R/\xi_0 > 0$, the saddle-point solution satisfies $\lambda \xi_0 q = i(R/\xi_0)^{1/2}$, reducing F_0 to the following form:

$$F_0 = 1 - \left(\frac{R}{\xi_0}\right)^{1/2}.$$

This is purely real, thereby rendering $F(t)$ in (16) a physical quantity and leading to the following PDF tail:

$$P(R; \mathbf{x}_0) \sim \exp\left\{-\frac{2}{3q}\left(\frac{R}{\xi_0}\right)^{3/2}\right\}. \quad (17)$$

Equation (17) provides the probability of finding a local Reynolds stress R, normalized by ξ_0, at $\mathbf{x} = \mathbf{x}_0$. Recall that $\xi_0 = -\partial_x \psi \partial_y \psi(\mathbf{x}_0)$ is local Reynolds stress associated with the modon solution (6). The PDF tail (17) is a stretched exponential, exhibiting non-Gaussian statistics and intermittency. Owing to this stretched exponential PDF tail, the probability of the generation of (large-scale) shear flow with a large amplitude is likely to be enhanced over Gaussian prediction. Note that in the absence of forcing (i.e., $\kappa_0 \to 0$), $P \to 0$, simply because the instanton cannot form without the forcing. We also note that an instanton method is likely to give an exponential dependence of PDF tails like Eq. (17).

Given a rather sensitive dependence of our results on the properties of an external forcing, it is worth commenting on the implication of our results for a real system where the stochasticity (randomness) is likely to be self-consistently generated. By virtue of the fluctuation-dissipation theorem, the amplitude of the (self-generated) fluctuation u' in a dynamical variable u may be crudely estimated as $\langle u'^2 \rangle / \tau_c \propto \langle f^2 \rangle$. Here, τ_c is the correlation time of u', which can represent the inverse of the linear growth rate of u', for instance. Thus, our results (given in terms of $\langle f^2 \rangle$) can be recast in terms of $\langle u'^2 \rangle / \tau_c$. We note though that in a self-consistent system, multiplicative, in addition to additive, stochasticity is likely to be present. In that case, the generation of coherent structures from this multiplicative noise may be different from the case of an external forcing. This issue will be addressed in a future publication.

In summary, we have presented a nonperturbative analytical result for the tail of the PDF of the local Reynolds stress (vorticity flux) R for forced Hasegawa-Mima turbulence. Our key idea was based on the observation that the PDF tail is governed by the bursty events associated with the appearance of coherent structures, which enabled us to relate the bursty events to the creation of a coherent structure, say, a modon in the present Letter. Accordingly, we envisioned the PDF tails as the transition amplitude from an initial state, with no fluid motion, to final states governed by the modon with different values of R in the long time limit. This transition amplitude was found by optimizing the effective action for the PDF given by a path integral via a saddle-point solution (instanton). We found that the tail of the Reynolds stress (R) PDF exhibits the non-Gaussian statistics with the specific form $\exp[-cR^{3/2}]$, where c is a constant. We conclude by remarking that the contribution to PDF tails from multi-structures or multi-instantons should be incorporated in a future paper. Furthermore, the extension of this model to the study of global momentum flux (Reynolds stress) and to the prediction of heat or particle transport in a more complicated model, such as the Hasegawa-Wakatani equations [8], ion temperature gradient turbulence [9], or dissipative trapped ion convective cell [10] models, would be worthwhile.

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